Lecture 12

14.2 - Limits and Continuity

Limits of functions of several variables are notoriously difficult to compute in general. First we give the definition, then we will say more:

Def: Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). We say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$. In this case, we write

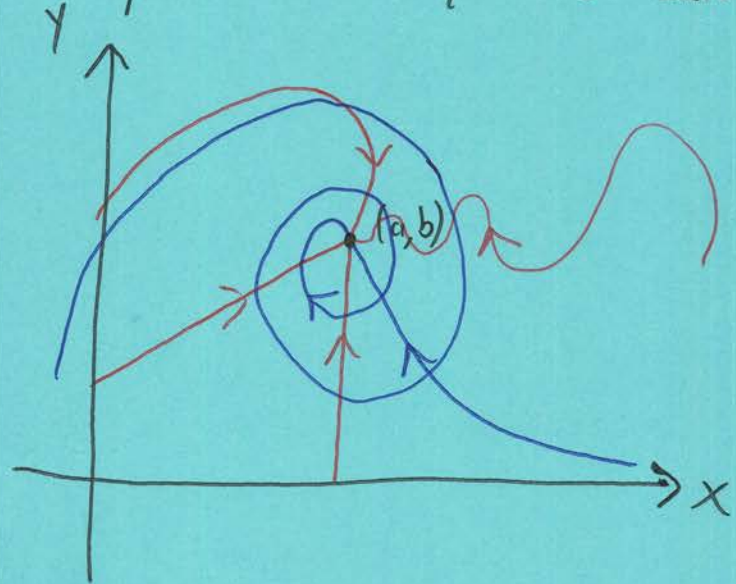
$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Intuitively, this means as your points get closer to (a,b) (coming from any direction!) the values f takes on is "squeezed towards L".

So, why are these limits notoriously difficult?

Back in Calc I, you said $\lim\limits_{x\to a} f(x)$ exists if $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$. There was only ~~two~~ ways to get to a: from the left and right. Here, in Calc III, we have <u>infinitely many ways</u> to get to a point (a,b), and they don't even have to be straight line!

Here are examples of ways we can get to (a,b):



It's much easier to show limits do not exist. Here's a useful rule for determining whether limits exist:

<u>Rule</u>: If $f(x,y) \to L_1$ as $(x,y)\to(a,b)$ along a path C_1 and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$.

<u>Ex</u>: Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ exist?

<u>Sol</u>: Checking along the x-axis, we use the path $(x,0)$, $x\to 0$.

$$\lim_{(x,0)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

and along the y-axis,

$$\lim_{(0,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

So, the limit does not exist. ◇

Be weary though, this doesn't always work! Sometimes we need paths other than the axes:

<u>Ex</u>: Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ exist?

<u>Sol</u>: along x-axis:

$$\lim_{(x,0)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = 0$$

along y-axis:

$$\lim_{(0,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = 0$$

but, along $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$ limit DNE. ◇

One useful trick for checking if limits <u>DO NOT</u> exist is to fix an arbitrary slope m and check along $y=mx$. (This is for $(x,y)\to(0,0)$, use $y=m(x-a)+b$ for $(x,y)\to(a,b)$)
If the limit depends on m, the limit does not exist.

<u>Ex</u>: Consider again $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.
Along the path $y=mx$, we have

$$\lim_{(x,mx)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{x^2+mx^2} = \frac{m}{1+m} \quad \therefore \text{depends on } m.$$

Again! Be weary:

<u>Ex</u>: Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist?

along $y=mx$

$$\lim_{(x,mx)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0$$

but, along $x=y^2$

$$\lim_{(y^2,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2} \neq 0.$$

So, the limit does not exist. $\diamond$

When can we expect to take limits?

<u>Def</u>: A function $f(x,y)$ is <u>continuous at (a,b)</u> if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. We say f is <u>continuous on D</u> if it is continuous at every point (a,b) in D.

<u>Facts</u>:
- Polynomials are continuous
- Sums of continuous functions are continuous
- Products of continuous functions are continuous
- Quotients of continuous functions are continuous wherever the denominator is $\neq 0$. Maybe even at these points, but it's subtle.
- Composition of continuous functions are continuous.

Let's see an example where we can compute the limit:

<u>Ex</u>: Does $\lim_{(x,y)\to(2,1)} \frac{4-xy}{x^2+3y^2}$ exist? If so, what is it?

Notice $f(x,y) = \frac{4-xy}{x^2+3y^2}$ is continuous at $(2,1)$ since the top and bottom are poly's and the denominator is not zero at $(2,1)$.

So, $\lim_{(x,y)\to(2,1)} \frac{4-xy}{x^2+3y^2} = \frac{4-(2)(1)}{2^2+3(1)^2} = \frac{4-2}{4+3} = \frac{2}{7}$. $\diamond$

Ex: Determine the set of points where the function

$$g(x,y) = \ln(x^2+y^2-4) - \sqrt{y}$$

is continuous.

Sol: We can use the facts from earlier to do this piece by piece.

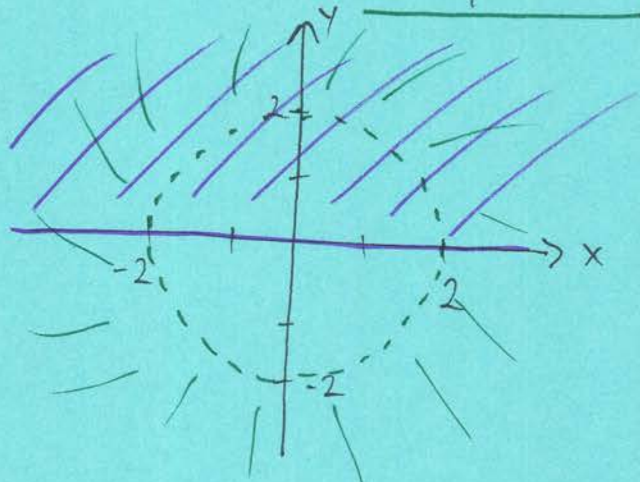
$\ln(x)$ is continuous when $x>0$, so

$\ln(x^2+y^2-4)$ is continuous when $x^2+y^2-4>0$

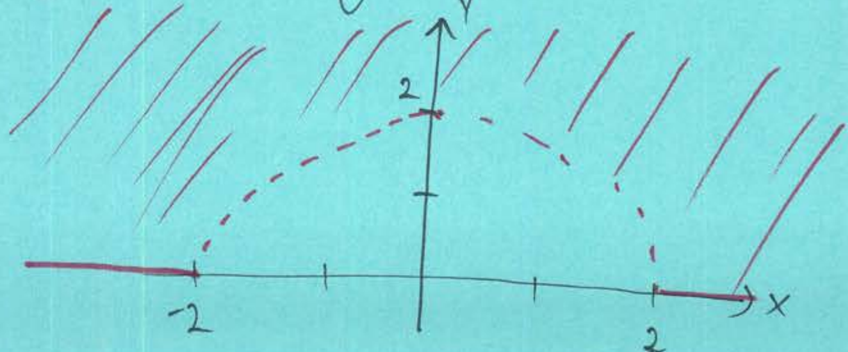
$\sqrt{y}$ is continuous for $y \geq 0$.

So g is continuous when $\underline{x^2+y^2-4>0}$ & $\underline{y\geq 0}$

Graphically:



So, the region where g is continuous is:



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